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LAPLACE TRANSFORMS OF POLYNOMIALLY BOUNDED VECTOR-VALUED FUNCTIONS AND SEMIGROUPS OF OPERATORS

BY

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ABSTRACT

For any natural number k, we characterize once-integrated Laplace transforms of $O((1+t)^k)$ and $O(t^k)$ Banach-space-valued functions. We use this to give Hille–Yosida type characterizations of generators of polynomially bounded strongly continuous semigroups, related families of operators, and solutions of the abstract Cauchy problem.

1. Introduction

There has been much interest recently in vector-valued Laplace transforms. Although Widder's theorem, for characterizing Laplace transforms of exponentially bounded Banach-space-valued functions, is not valid in general, an "integrated version," due to Arendt ([1]; see also [10]), is (see Lemma 2.5).

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In this paper, we give a similar characterization for once-integrated Laplace transforms of polynomially bounded Banach-space-valued functions (Theorem 2.6).

Part of the interest in vector-valued Laplace transforms arises from its connection with linear abstract Cauchy problems. We will restrict ourselves in this paper to the first-order abstract Cauchy problem

$$\text{(ACP)} \qquad \frac{d}{dt}u(t,x) = A(u(t,x)) \quad (t \geq 0), \quad u(0,x) = x,$$

although similar characterizations are possible for second-order abstract Cauchy problems, integrodifferential equations, etc. (see [6]).

As a corollary of our Laplace transform result, for any $k \in \mathbf{N}$, we will characterize $O((1+t)^k)$ solutions of (ACP) (Theorem 3.6) and generators of $O((1+t)^k)$ strongly continuous semigroups (Corollary 2.10), in terms of the resolvents of the generator, analogously to the Hille–Yosida–Phillips theorem. We similarly characterize densely defined generators of polynomially bounded integrated semigroups and regularized semigroups in terms of the resolvent (Theorems 4.8, 4.9 and 4.6). When the density of the domain is removed, these resolvent conditions are equivalent to the appropriate once-integrated family of operators $\{S(t)\}_{t\geq 0}$ being locally Lipschitz continuous, with Lipschitz constant growing like a polynomial, that is,

$$\overline{\lim}_{h\to 0^+} \frac{1}{h}\|S(t+h) - S(t)\| \leq M(1+t)^k,$$

for all nonnegative t (Theorems 2.9, 4.4, 4.6 and 4.8).

For arbitrary $k \in \mathbf{N}$, we give a much simpler sufficient condition, involving only the first power of the resolvent in the right half plane, for generating a $O((1+t)^k)$ $(1-A)^{-2}$-regularized semigroup; this produces mild $O((1+t)^k)$ solutions of (ACP), for all initial data x in the domain of A^2 (Theorem 4.10). This sufficient condition is close to a necessary condition (see Remark 4.11).

A simple example of a strongly continuous semigroup that is polynomially bounded but not bounded is translation on a weighted L^p space, $L^p(\mathbf{R}, \omega(x)dx)$,

$$(T(t)f)(x) \equiv f(x+t) \quad (x \in \mathbf{R}, t \geq 0),$$

for $1 \leq p < \infty$. It is not hard to see that

$$\|T(t)\|^p = \sup_{s\in\mathbf{R}} \frac{\omega(s-t)}{\omega(s)},$$

for any $t \geq 0$. For ω a nonnegative polynomial, $\|T(t)\|$ grows like a polynomial, as $t \to \infty$.

We also introduce, for any nonnegative integer k, maximal continuously embedded subspaces on which an operator generates a $O((1+t)^k)$ strongly continuous semigroup (Theorem 3.6).

Throughout this paper, X is a Banach space and A is a (possibly unbounded) linear operator on X, with domain $\mathcal{D}(A)$ and resolvent set $\rho(A)$. We will denote by $B(X)$ the space of all bounded linear operators from X to itself.

2. Polynomially bounded once-integrated Laplace transform

In this section we present our Laplace transform result (Theorem 2.6), and show how it immediately produces generation theorems for polynomially bounded strongly continuous semigroups (Corollary 2.10) and once-integrated semigroups (Theorem 2.9).

Definition 2.1: Suppose W is a Banach space. We will say that $\{g(s)\}_{s>0} \subseteq W$ is the **once-integrated Laplace transform of** G if G: $[0,\infty) \to W$ is continuous, $G(0)=0$, and

$$g(s) = s\int_0^\infty e^{-st}G(t)\,dt \quad (s>0).$$

We call G the **once-integrated determining function of** g, and say that g is a **once-integrated Laplace transform.**

LEMMA 2.2: *Suppose g: $(0,\infty) \to \mathbf{C}$ and $k \in \mathbf{N} \cup \{0\}$. Then the following are equivalent.*

(a) *There exists a constant M_1 so that g is the once-integrated Laplace transform of a continuous G: $[0,\infty) \to \mathbf{C}$ such that*

$$\overline{\lim}_{h\to 0^+} \frac{1}{h}|G(t+h)-G(t)| \leq M_1(1+t^k), \quad \forall t \geq 0.$$

(b) *There exists a constant M_2 so that g is infinitely differentiable and*

$$|g^{(n)}(s)| \leq M_2\left[\frac{n!}{s^{n+1}} + \frac{(n+k)!}{s^{n+k+1}}\right], \quad \forall s>0, \quad n \in \mathbf{N}\cup\{0\}.$$

M_1 may be chosen so that $M_1 \leq 2M_2$.

Proof: (a) $\rightarrow$ (b) is clear. For (b) $\rightarrow$ (a), first assume that g is real valued. (b) implies that

$$h_{\pm}(s) \equiv M_2 \left(\frac{1}{s} + \frac{k!}{s^{k+1}} \right) \pm g(s)$$

is completely monotone; that is,

$$(-1)^n h_{\pm}^{(n)}(s) \geq 0, \quad \forall s > 0, \quad n \in \mathbf{N} \cup \{0\}.$$

By Bernstein's theorem ([22, page 156, Corollary 7]), there exist nondecreasing $\alpha_{\pm}$ such that

$$h_{\pm}(s) = \int_0^{\infty} e^{-st}\, d\alpha_{\pm}(t) \quad (s > 0),$$

so that

$$(*) \qquad \begin{aligned} g(s) &= s \int_0^{\infty} e^{-st}\, d\left(\alpha_+(t) - M_2 \left(t + \frac{t^{k+1}}{k+1} \right) \right) \\ &= s \int_0^{\infty} e^{-st}\, d\left(M_2 \left(t + \frac{t^{k+1}}{k+1} \right) - \alpha_-(t) \right), \end{aligned}$$

for $s > 0$. By the uniqueness of the Laplace transform and $(*)$,

$$(**) \qquad G(t) \equiv \left(\alpha_+(t) - M_2 \left(t + \frac{t^{k+1}}{k+1} \right) \right) = \left(M_2 \left(t + \frac{t^{k+1}}{k+1} \right) - \alpha_-(t) \right).$$

We have shown that g is the once-integrated Laplace transform of G. By $(**)$,

$$\overline{\lim}_{h \to 0^+} \frac{1}{h} |G(t+h) - G(t)| \leq M_2(1 + t^k), \quad \forall t \geq 0.$$

If g is complex-valued, with real part g_1 and imaginary part g_2, then, for $j = 1, 2$, let G_j be the once-integrated determining function for g_j. We have shown that

$$\overline{\lim}_{h \to 0^+} \frac{1}{h} |G_j(t+h) - G_j(t)| \leq M_2(1 + t^k), \quad \forall t \geq 0.$$

Define $G \equiv G_1 + iG_2$, then g is the once-integrated Laplace transform of G and

$$\overline{\lim}_{h \to 0^+} \frac{1}{h} |G(t+h) - G(t)| \leq 2M_2(1 + t^k), \quad \forall t \geq 0,$$

as desired. ∎

The same proof shows the following.

LEMMA 2.3: *Suppose* $g\colon (0,\infty) \to \mathbf{C}$ *and* $k \in \mathbf{N} \cup \{0\}$. *Then the following are equivalent.*

(a) *There exists a constant* M_1 *so that* g *is the once-integrated Laplace transform of a continuous* $G\colon [0,\infty) \to \mathbf{C}$ *such that*

$$\overline{\lim}_{h\to 0^+} \frac{1}{h}|G(t+h) - G(t)| \le M_1 t^k, \quad \forall t \ge 0.$$

(b) *There exists a constant* M_2 *so that* g *is infinitely differentiable and*

$$|g^{(n)}(s)| \le M_2 \frac{(n+k)!}{s^{n+k+1}}, \quad \forall s > 0, \quad n \in \mathbf{N} \cup \{0\}.$$

M_1 *may be chosen so that* $M_1 \le 2M_2$.

LEMMA 2.4: *Let* $k \in \mathbf{N}$. *For every* $0 < \alpha < k$, *the following holds.*

$$(1+|s|)^k \le K_\alpha e^{\alpha|s|}, \quad \forall s \in \mathbf{R},$$

where $K_\alpha \equiv (\frac{k}{\alpha})^k e^{-k+\alpha}$.

Proof: The lemma follows from the evident equality

$$\max_{s\in\mathbf{R}} (1+|s|)^k e^{-\alpha|s|} = \left(\frac{k}{\alpha}\right)^k e^{-k+\alpha}. \quad \blacksquare$$

LEMMA 2.5 ([1, Theorem 1.1]): *Suppose* W *is a Banach space,* $M, \alpha > 0$, *and* $g\colon (0,\infty) \to W$. *Then the following are equivalent.*

(a) g *is the once-integrated Laplace transform of a continuous function* G *satisfying*

$$\overline{\lim}_{h\to 0^+} \frac{1}{h}\|G(t+h) - G(t)\| \le M e^{\alpha t} \quad (t \ge 0).$$

(b) g *is infinitely differentiable and for* $n \in \mathbf{N} \cup \{0\}$,

$$\left\|\frac{(s-\alpha)^{n+1}}{n!} g^{(n)}(s)\right\| \le M, \quad \forall s > \alpha.$$

THEOREM 2.6: *Suppose W is a Banach space, $k \in \mathbf{N}$ and g: $(0,\infty) \to W$. Then the following are equivalent.*

(a) *There exists a constant M_1 so that g is the once-integrated Laplace transform of a continuous function G satisfying*

$$\overline{\lim}_{h\to 0^+} \frac{1}{h}\|G(t+h)-G(t)\| \leq M_1(1+t)^k, \quad \forall t \geq 0.$$

(b) *There exists a constant M_2 so that g is infinitely differentiable and*

$$\|g^{(n)}(s)\| \leq M_2\left[\frac{n!}{s^{n+1}}+\frac{(n+k)!}{s^{n+k+1}}\right], \quad \forall s>0, \quad n \in \mathbf{N}\cup\{0\}.$$

(c) *There exists a constant M_1 so that g is infinitely differentiable and for $0<\alpha<k$, $n \in \mathbf{N}\cup\{0\}$,*

$$\left\|\frac{(s-\alpha)^{n+1}}{n!}g^{(n)}(s)\right\| \leq M_1\left(\frac{k}{\alpha}\right)^k e^{-k+\alpha}, \quad \forall s>\alpha.$$

Proof: (b) $\to$ (a). By Lemma 2.2, for any $x^* \in W^*$, there exist continuous G_{x^*}: $[0,\infty) \to \mathbf{C}$ such that $s \mapsto < g(s), x^* >$ is the once-integrated Laplace transform of G_{x^*}, and

$$\overline{\lim}_{h\to 0^+} \frac{1}{h}|G_{x^*}(t+h)-G_{x^*}(t)| \leq 2M_2\|x^*\|(1+t^k), \quad \forall t \geq 0.$$

By Lemma 2.5 and its proof (see [1, Theorem 1.1]), this is equivalent to the existence of G: $[0,\infty) \to W$ such that

$$G_{x^*}(t) =< G(t), x^* >, \quad \forall t \geq 0, \quad x^* \in W^*.$$

G is clearly the desired family of (a).

(a) $\to$ (b) is clear, since there exists M_2 so that

$$M_1(1+t)^k \leq M_2(1+t^k), \quad \forall t \geq 0.$$

(a) $\to$ (c). From Lemma 2.4,

$$\overline{\lim}_{h\to 0^+} \frac{1}{h}\|G(t+h)-G(t)\| \leq M_1K_\alpha e^{\alpha t}, \quad \forall \alpha>0, \quad t \geq 0, \tag{2.1}$$

where K_α is defined in Lemma 2.4. Hence (c) is true by Lemma 2.5 with M replaced by M_1K_α.

(c) $\to$ (a). From Lemma 2.5, (c) implies (2.1). Since α and t are independent, we may choose $\alpha = k(1+t)^{-1}$ in (2.1) to obtain

$$\overline{\lim}_{h\to 0^+}\frac{1}{h}\|G(t+h)-G(t)\| \le M_1\left(\frac{k}{k(1+t)^{-1}}\right)^k e^{-k+k(1+t)^{-1}}e^{k(1+t)^{-1}t}$$
$$= M_1(1+t)^k, \quad \forall t \ge 0. \qquad \blacksquare$$

The same proof, with Lemma 2.2 replaced by Lemma 2.3, gives us the following.

THEOREM 2.7: *Suppose W is a Banach space, $g\colon (0,\infty) \to W$ and $k \in \mathbf{N}$. Then the following are equivalent.*

(a) *There exists a constant M_1 so that g is the once-integrated Laplace transform of a continuous G such that*
$$\overline{\lim}_{h\to 0^+}\frac{1}{h}\|G(t+h)-G(t)\| \le M_1 t^k, \quad \forall t \ge 0.$$

(b) *There exists a constant M_2 so that g is infinitely differentiable and*
$$\|g^{(n)}(s)\| \le M_2\frac{(n+k)!}{s^{n+k+1}}, \quad \forall s > 0, \quad n \in \mathbf{N}\cup\{0\}.$$

Definition 2.8: The exponentially bounded strongly continuous family of operators $\{S(t)\}_{t\ge 0}$ is a **once-integrated semigroup, generated by** A if $S(0)=0$ and there exists real α such that $(\alpha,\infty)\subseteq \rho(A)$ and
$$(s-A)^{-1}x = s\int_0^\infty e^{-st}S(t)x\,dt \quad (x\in X, s>\alpha).$$

THEOREM 2.9: *The following are equivalent, if $k \in \mathbf{N}$.*

(a) *There exists a constant M_1 so that A generates a once-integrated semigroup $S(t)$ such that*
$$\overline{\lim}_{h\to 0^+}\frac{1}{h}\|S(t+h)-S(t)\| \le M_1(1+t)^k,$$
for all $t \ge 0$.

(b) *There exists a constant M_2 so that $(0,\infty)\subseteq\rho(A)$ and for $n \in \mathbf{N}$,*
$$\|s^n(s-A)^{-n}\| \le M_2\left[1+\frac{(n+k-1)!}{(n-1)!}s^{-k}\right], \quad \forall s>0.$$

(c) *There exists a constant M_3 so that $(0,\infty)\subseteq\rho(A)$ and for $0<\alpha<k$, $n\in\mathbf{N}$,*
$$\|(s-\alpha)^n(s-A)^{-n}\| \le M_3\left(\frac{k}{\alpha}\right)^k e^{-k+\alpha}, \quad \forall s>\alpha.$$

Proof: This is immediate from the definition of once-integrated semigroup and Theorem 2.6, when one notes that

$$\left(\frac{d}{ds}\right)^n (s-A)^{-1} = (-1)^n n!(s-A)^{-(n+1)},$$

for any nonnegative integer n, and $s > 0$. ∎

COROLLARY 2.10: *The following are equivalent, if $\mathcal{D}(A)$ is dense, and $k \in \mathbf{N}$.*

(a) *There exists a constant M_1 so that A generates a strongly continuous semigroup $T(t)$ such that*

$$\|T(t)\| \le M_1(1+t)^k,$$

for all $t \ge 0$.

(b) *There exists a constant M_2 so that $(0,\infty) \subseteq \rho(A)$ and for $n \in \mathbf{N}$,*

$$\|s^n(s-A)^{-n}\| \le M_2\left[1 + \frac{(n+k-1)!}{(n-1)!}s^{-k}\right], \quad \forall s > 0.$$

(c) *There exists a constant M_3 so that $(0,\infty) \subseteq \rho(A)$ and for $0 < \alpha < k$, $n \in \mathbf{N}$,*

$$\|(s-\alpha)^n(s-A)^{-n}\| \le M_3\left(\frac{k}{\alpha}\right)^k e^{-k+\alpha}, \quad \forall s > \alpha.$$

Proof: In [1, Corollary 4.2] it is shown that, when $\mathcal{D}(A)$ is dense, then A generates a strongly continuous semigroup $\{T(t)\}_{t\ge0}$ if and only if A generates a locally Lipschitz continuous once-integrated semigroup $\{S(t)\}_{t\ge0}$ satisfying, for some $K, \alpha > 0$,

$$\overline{\lim}_{h\to0^+}\frac{1}{h}\|S(t+h) - S(t)\| \le Ke^{\alpha t}, \quad \forall t \ge 0,$$

with

$$S(t)x = \int_0^t T(r)x\,dr \quad (x \in X, t \ge 0).$$

Thus (a) of this Corollary is clearly equivalent to (a) of Theorem 2.9. ∎

3. Maximal subspaces for polynomially bounded solutions of the abstract Cauchy problem

Maximal continuously embedded subspaces where a closed operator generates a strongly continuous semigroup of contractions are constructed, independently, in [14] and [15].

In this section, we will use a slight modification of the construction in [5, Chapter V] to produce, for any nonnegative integer k, maximal continuously embedded subspaces where a closed operator generates a $O((1+t)^k)$ strongly continuous semigroup, or a once-integrated semigroup as in Theorem 2.9(a). For the latter, we may give Hille–Yosida type conditions on the resolvent at a point x that characterize x being in the subspace; in other words, we are characterizing $O((1+t)^k)$ solutions of (ACP).

Assume throughout this section that A is closed, $\alpha \in \mathbf{R}$, $(s-A)$ is injective, for $s > \alpha$, and k and m are nonnegative integers. For $k = 0$, the results of this section are in [5, Chapter V].

Definition 3.1: A **mild solution** of (ACP) is u such that $t \mapsto u(t,x) \in C([0,\infty), X)$, $\int_0^t u(r,x)\,dr \in \mathcal{D}(A)$, for all $t \geq 0$, and

$$u(t,x) = A\left(\int_0^t u(r,x)\,dr\right) + x \quad (t \geq 0).$$

A closed operator A generates a strongly continuous semigroup if and only if (ACP) has a unique mild solution for all $x \in X$ (see [5, Corollary 4.11] or [19, Theorem 3.1]). This automatically implies that A is densely defined. For operators A that are not densely defined, it becomes natural to consider a weaker definition of solution.

Definition 3.2: A **weak mild** solution of (ACP) is v such that $t \mapsto v(t,x)$ is locally Lipschitz continuous, $\int_0^t v(r,x)\,dr \in \mathcal{D}(A)$, for all $t \geq 0$, and

$$v(t,x) = A\left(\int_0^t v(r,x)\,dr\right) + tx \quad (t \geq 0).$$

We justify this terminology by noting that, when $\mathcal{D}(A)$ is dense, so that A^* is defined, then

$$\frac{d}{dt}\langle v(t,x), x^*\rangle = \langle v(t,x), A^*x^*\rangle + \langle x, x^*\rangle$$

for $x^* \in \mathcal{D}(A^*)$, almost all $t \geq 0$; see [3] or [19, Chapter 3], for the definition of a **weak solution** of (ACP).

Both these notions of solution, when exponentially bounded, may be equated with the Laplace transform. The following result, from [11], is actually stated in a version that does not assume $(s-A)$ is injective.

LEMMA 3.3 ([11, Theorem 2.1]): *Suppose* $t \mapsto w(t) \in C([0,\infty),X)$ *and is* $O(e^{\alpha t})$. *Then the following are equivalent.*

(a) $(s-A)^{-1}x = s^m \int_0^\infty e^{-st} w(t)\, dt \quad (s > \alpha)$.

(b) *For all* $t \geq 0$, $\int_0^t w(s)\, ds \in \mathcal{D}(A)$, *with*

$$w(t) = A\left(\int_0^t w(s)\, ds\right) + \frac{t^m}{m!}x.$$

As an immediate consequence of [5, Lemma 2.10], we have the following.

LEMMA 3.4: *Any mild or weak mild exponentially bounded solution of (ACP) is unique.*

Definition 3.5: Let $Z(A,k)$ be the set of all x for which (ACP) has a mild solution such that

$$t \mapsto (1+t)^{-k} u(t,x)$$

is uniformly continuous and bounded on $[0,\infty)$. Define

$$\|x\|_{Z(A,k)} \equiv \sup_{t\geq 0}(1+t)^{-k}\|u(t,x)\|.$$

Let $Y(A,k)$ be the set of all x for which (ACP) has a weak mild solution such that

$$\text{ess sup}\{(1+t)^{-k}|\frac{d}{dt} < v(t,x), x^* > |\,|\, t \geq 0, x^* \in X^*, \|x^*\| \leq 1\} < \infty.$$

Define

$$\|x\|_{Y(A,k)} \equiv \text{ess} \sup_{t\geq 0, x^* \in X^*, \|x^*\|\leq 1} \{(1+t)^{-k}|\frac{d}{dt} < v(t,x), x^* > |, | < x, x^* > |\}.$$

Then almost exactly as in the proof of [5, Theorems 5.5 and 5.10], we have the following, where, for W continuously embedded in X, we write $A|_W$ for the restriction of A to W; that is, $\mathcal{D}(A|_W) \equiv \{x \in W \cap \mathcal{D}(A) \colon Ax \in W\}$.

THEOREM 3.6:

(1) *$Y(A,k)$ and $Z(A,k)$ are Banach spaces continuously embedded in X.*

(2) *$Z(A,k)$ equals the closure, in $Y(A,k)$, of $\mathcal{D}(A|_{Y(A,k)})$.*

(3) *$A|_{Z(A,k)}$ generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$, on $Z(A,k)$, such that*

$$\|T(t)\| \leq (1+t)^k, \ \forall t \geq 0.$$

(4) *$A|_{Y(A,k)}$ generates a once-integrated semigroup $\{S(t)\}_{t\geq 0}$ such that*

$$\overline{\lim}_{h\to 0^+} \frac{1}{h}\|S(t+h) - S(t)\| \leq (1+t)^k, \quad \forall t \geq 0.$$

(5) *If $k \in \mathbf{N}$, then $x \in Y(A,k)$ if and only if $x \in \mathrm{Im}((s-A)^n)$, for all $n \in \mathbf{N}, s > 0$, and there exists $M > 0$ so that, for $0 < \alpha < k$, $n \in \mathbf{N}$,*

$$\|(s-\alpha)^n (s-A)^{-n} x\| \leq M \left(\frac{k}{\alpha}\right)^k e^{-k+\alpha}, \quad \forall s > \alpha.$$

(6) *$x \in Y(A,k)$ if and only if there exists $M > 0$ such that, for all $n \in \mathbf{N}$, $s > 0$, we have $x \in \mathrm{Im}((s-A)^n)$, with*

$$\|s^n (s-A)^{-n} x\| \leq M \left[1 + \frac{(n+k-1)!}{(n-1)!} s^{-k}\right], \quad \forall s > 0.$$

(7) *$Z(A,k)$ is maximal unique; that is, if W is a Banach space continuously embedded in X such that $A|_W$ generates a strongly continuous $O((1+t)^k)$ semigroup, then W is continuously embedded in $Z(A,k)$.*

Proof: The proofs of assertions (1), (3) and (7) are the same as the proofs of [5, Theorem 5.5(1),(2),(5) and (8)].

As in [5, Theorem 5.5(4)], we may show that $A|_{Y(A,k)}$ satisfies the resolvent conditions in (b) of Theorem 2.9, so that the conclusion of (4) follows from Theorem 2.9.

Similarly, if W is the closure of $\mathcal{D}(A|_{Y(A,k)})$, in $Y(A,k)$, then because $A|_W$ satisfies the resolvent conditions in (b) of Corollary 2.10, $A|_W$ generates a $O((1+t)^k)$ strongly continuous semigroup. By (7), W is continuously embedded in $Z(A,k)$. But $Z(A,k) \subseteq Y(A,k)$, and by (3), the domain of $\mathcal{D}(A|_{Z(A,k)}) \subseteq \mathcal{D}(A|_{Y(A,k)})$ is dense in $Z(A,k)$, so $Z(A,k) \subseteq W$. This gives us (2).

Assertions (5) and (6) follow from Lemma 3.3 and Theorem 2.6, as in the proof of [5, Theorem 5.10]. ∎

4. Polynomially bounded existence families and integrated existence families

When solutions of (ACP) exist for some, but not all, initial data, an alternative to going to a continuously embedded subspace, as in the previous section, is to construct bounded operators on the original space that produce the solutions (see [5]). In this section, for k a nonnegative integer, we discuss when such families of operators exist and are $O((1+t)^k)$.

Throughout this section, $C \in B(X)$.

Definition 4.1: [5, 17]. Suppose $m \in \mathbf{N} \cup \{0\}$ and A is closable. The strongly continuous family of bounded operators $\{S(t)\}_{t\geq 0}$ is a **mild** m**-times integrated** C**-existence family for** A if, for all $x \in X$, $t \geq 0$, $\int_0^t S(s)x\,ds \in \mathcal{D}(A)$, and

$$A\left(\int_0^t S(s)x\,ds\right) = S(t)x - \frac{t^m}{m!}Cx.$$

If $m = 0$, we simply call $\{S(t)\}_{t\geq 0}$ a mild C-existence family for A ([5]).

If $C = I$, then $\{S(t)\}_{t\geq 0}$ is an m-times integrated semigroup for A. In this case, A is unique, and is called the **generator** of $\{S(t)\}_{t\geq 0}$.

Definition 4.2: The complex number s is in $\rho_C(A)$, the C**-regularized resolvent** of A, if $(s - A)$ is injective and $\mathrm{Im}(C) \subseteq \mathrm{Im}(s - A)$.

As a consequence of Lemma 3.3, we have the following.

PROPOSITION 4.3: *Suppose $m \in \mathbf{N} \cup \{0\}$, $\alpha > 0$, A is closed, $(s - A)$ is injective, for $s > \alpha$ and $\{S(t)\}_{t\geq 0}$ is a $O(e^{\alpha t})$ strongly continuous family of bounded operators. Then the following are equivalent.*

(a) *$\{S(t)\}_{t\geq 0}$ is a mild m-times integrated C-existence family for A.*
(b) *$(\alpha, \infty) \subseteq \rho_C(A)$ and*

$$(s - A)^{-1}Cx = s^m \int_0^\infty e^{-st} S(t)x\,dt$$

for all $x \in X, s > \alpha$.

THEOREM 4.4: *Suppose A is closed, $k \in \mathbf{N}$, and $(s - A)$ is injective, for all $s > 0$. Then the following are equivalent.*

(a) *There exists a mild once-integrated C-existence family $\{S(t)\}_{t\geq 0}$ for A such that, for some constant M_1,*

$$\overline{\lim}_{h\to 0^+} \frac{1}{h}\|S(t+h) - S(t)\| \leq M_1(1+t)^k \ \forall t \geq 0.$$

(b) *There exists a constant* M_2 *so that* $\mathrm{Im}(C) \subseteq \mathrm{Im}((s-A)^n)$, *for all* $s > 0$, $n \in \mathbf{N}$, *with*

$$\|s^n(s-A)^{-n}C\| \le M_2 \left[1 + \frac{(n+k-1)!}{(n-1)!} s^{-k}\right], \ \forall s > 0.$$

(c) *There exists a constant* M_3 *so that* $\mathrm{Im}(C) \subseteq \mathrm{Im}((s-A)^n)$, *for all* $n \in \mathbf{N}$, $s > 0$, *and for* $0 < \alpha < k$, $n \in \mathbf{N}$,

$$\|(s-\alpha)^n(s-A)^{-n}C\| \le M_3 \left(\frac{k}{\alpha}\right)^k e^{-k+\alpha}, \quad \forall s > \alpha.$$

Proof: (c) → (a). By [5, Lemma 5.23], $s \mapsto (s-A)^{-1}C$, from $(0,\infty) \to B(X)$, is differentiable, with

$$(*) \qquad \left(\frac{d}{ds}\right)^n ((s-A)^{-1}C) = (-1)^n n!(s-A)^{-(n+1)},$$

for all nonnegative integers n. Theorem 2.6 and Proposition 4.3 now give us (a).

(a) → (c). By Theorem 2.6 and Proposition 4.3, $s \mapsto (s-A)^{-1}C$, from $(0,\infty) \to B(X)$, is infinitely differentiable, and

$$\|(s-\alpha)^{n+1}(\frac{d}{ds})^n ((s-A)^{-1}C)\| \le Mn!(\frac{k}{\alpha})^k e^{-k+\alpha}, \quad \forall s > \alpha, \quad n \in \mathbf{N} \cup \{0\}.$$

An induction argument shows that $\mathrm{Im}(C) \subseteq \mathrm{Im}((s-A)^n)$ and (∗) holds, for any nonnegative integer n. This gives us (c).

The equivalence of (a) and (b) is shown identically. ∎

Definition 4.5: [16], [17] and [21]. Suppose $m \in \mathbf{N}$. The strongly continuous family of operators $\{S(t)\}_{t\ge 0} \subseteq B(X)$ is an m-times integrated C-regularized semigroup if C is injective, $S(0) = 0$ and

$$S(t)S(s)x = \left[\int_s^{t+s} - \int_0^t\right] (t+s-r)^{m-1} S(r)Cx\, dr,$$

for all $x \in X, s, t \ge 0$.

The strongly continuous family $\{W(t)\}_{t\ge 0} \subseteq B(X)$ is a C-**regularized semigroup** if $W(t)W(s) = CW(t+s)$, for all $s, t \ge 0$, and $W(0) = C$ is injective. A 0-times integrated C-regularized semigroup will be a C-regularized semigroup.

For $m \in \mathbf{N} \cup \{0\}$, A closable, we will say that the m-times integrated C-regularized semigroup is an m-times integrated C-regularized semigroup for A

if $S(t)A \subseteq AS(t)$, for all $t \geq 0$, and $\{S(t)\}_{t\geq 0}$ is a mild m-times integrated C-existence family for A (in [17], A is called a **subgenerator** of $\{S(t)\}_{t\geq 0}$). We will also say that A has an m-times integrated C-regularized semigroup or has the m-times integrated C-regularized semigroup.

For $C \neq I$, even when $m = 0$, A is not unique, even if we insist that A be closed; that is, given a C-regularized semigroup $\{W(t)\}_{t\geq 0}$, there may be more than one closed operator A for which $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup (see [7, Counterexample 0.2]).

The operator A **generates** a C-regularized semigroup $\{W(t)\}_{t\geq 0}$ if

$$Ax = C^{-1}\left[\lim_{t\to 0}\frac{1}{t}(W(t)x - Cx)\right],$$

with maximal domain. The generator is the maximal A such that $\{W(t)\}_{t\geq 0}$ is a C-regularized semigroup for A.

THEOREM 4.6: *Suppose A is closed, C is injective, and $k \in \mathbf{N}$. Then the following are equivalent.*

(a) *There exists a once-integrated C-regularized semigroup $\{S(t)\}_{t\geq 0}$ for A such that, for some constant M_1,*

$$\overline{\lim}_{h\to 0^+}\frac{1}{h}\|S(t+h) - S(t)\| \leq M_1(1+t)^k \quad \forall t \geq 0.$$

(b) *$CA \subseteq AC$, $(0,\infty) \subseteq \rho_C(A)$, and there exists a constant M_2 so that $\mathrm{Im}(C) \subseteq \mathrm{Im}((s-A)^n)$, for all $s > 0$, $n \in \mathbf{N}$, with*

$$\|s^n(s-A)^{-n}C\| \leq M_2\left[1 + \frac{(n+k-1)!}{(n-1)!}s^{-k}\right], \quad \forall s > 0.$$

(c) *$CA \subseteq AC$, $(0,\infty) \subseteq \rho_C(A)$, $\mathrm{Im}(C) \subseteq \mathrm{Im}((s-A)^n)$, for all $n \in \mathbf{N}, s > 0$ and there exists a constant M_3 so that for $0 < \alpha < k$, $n \in \mathbf{N}$,*

$$\|(s-\alpha)^n(s-A)^{-n}C\| \leq M_3\left(\frac{k}{\alpha}\right)^k e^{-k+\alpha}, \quad \forall s > \alpha.$$

If $\mathcal{D}(A)$ is dense, then these are equivalent to the following.

(d) *There exists a C-regularized semigroup $\{W(t)\}_{t\geq 0}$ for A such that, for some constant M_1,*

$$\|W(t)\| \leq M_1(1+t)^k \quad \forall t \geq 0.$$

Proof: (a) $\rightarrow$ (b). By [17, Lemma 5.2], $CA \subseteq AC$ and $(s - A)$ is injective, for all $s > 0$. Thus (b) follows from Theorem 4.4.

(b) $\rightarrow$ (a). Let $\{S(t)\}_{t\geq 0}$ be as in (a) of Theorem 4.4. Since $CA \subseteq AC$, it follows from Proposition 4.3 and [16] or [17, Lemma 4.1] that $\{S(t)\}_{t\geq 0}$ is a once-integrated C-regularized semigroup for A.

The equivalence of (a) and (c) follows identically.

When $\mathcal{D}(A)$ is dense, the equivalence of (a) and (d) follows from [21, Lemma 5.1], and its proof. ∎

Remarks 4.7: When $\rho(A)$ is nonempty, then whenever $\{S(t)\}_{t\geq 0}$ is a once-integrated C-regularized semigroup for A, it is not hard to show that, for any $s \in \rho(A)$,

$$W_s(t) \equiv \frac{d}{dt} S(t)(s - A)^{-1} \quad (t \geq 0)$$

is an $(s - A)^{-1}C$-regularized semigroup for A.

The obvious analogue of Theorem 4.6, for groups, is also true; replace "semigroup" by "group", "t nonnegative" by "t real," and "$s > 0$ (α)" by "$|s| > 0$ (α)," in Theorem 4.6.

THEOREM 4.8: *Suppose $k \in \mathbf{N}$ and m is a nonnegative integer. Then the following are equivalent.*

(a) *There exists a constant M_1 so that A is the generator of an $(m+1)$-times integrated semigroup $\{S(t)\}_{t\geq 0}$ satisfying*

$$\overline{\lim}_{h\to 0^+} \frac{1}{h} \|S(t + h) - S(t)\| \leq M_1(1 + t)^k, \quad \forall t \geq 0.$$

(b) *There exists a constant M_2 so that $(0, \infty) \subseteq \rho(A)$ and for $n \in \mathbf{N} \cup \{0\}$,*

$$\left\| \left[\frac{(s - A)^{-1}}{s^m} \right]^{(n)} \right\| \leq M_2 \left[\frac{n!}{s^n} + \frac{(n + k)!}{s^{n+k+1}} \right], \quad \forall s > 0.$$

(c) *There exists a constant M_3 so that $(0, \infty) \subseteq \rho(A)$ and for $0 < \alpha < k$, $n \in \mathbf{N} \cup \{0\}$,*

$$\left\| (s - \alpha)^{n+1} \left[\frac{(s - A)^{-1}}{s^m} \right]^{(n)} \right\| \leq M_3 \left(\frac{k}{\alpha} \right)^k e^{-k+\alpha} n!, \quad \forall s > \alpha.$$

If $\mathcal{D}(A)$ is dense, then these are equivalent to the following.

(d) *There exists a constant M_1 so that A is the generator of an m-times integrated semigroup $\{W(t)\}_{t\geq 0}$ such that*

$$\|W(t)\| \leq M_1(1+t)^k, \quad \forall t \geq 0.$$

Proof: The equivalence of (a), (b) and (c) is immediate from Theorem 2.6. When $\mathcal{D}(A)$ is dense, the equivalence of (a) and (d) follows from [1, Corollary 4.2]. ∎

In the following, we give equivalent conditions for being the generator of a particularly desirable class of integrated semigroups, that appear often in practice, k-times integrated semigroups that are $O(t^k)$; see [2], [8], [12] and [13], for example.

THEOREM 4.9: *Suppose $k \in \mathbf{N}$. Then the following are equivalent.*

(a) *There exists a constant M_1 so that A is the generator of a $(k+1)$-times integrated semigroup $\{S(t)\}_{t\geq 0}$ satisfying*

$$\overline{\lim}_{h\to 0^+} \frac{1}{h}\|S(t+h) - S(t)\| \leq M_1 t^k, \quad \forall t \geq 0.$$

(b) *There exists a constant M_2 so that $(0,\infty) \subseteq \rho(A)$ and for $n \in \mathbf{N} \cup \{0\}$,*

$$\left\| \left[\frac{(s-A)^{-1}}{s^k} \right]^{(n)} \right\| \leq M_2 \frac{(n+k)!}{s^{n+k+1}}, \quad \forall s > 0.$$

If $\mathcal{D}(A)$ is dense, then these are equivalent to the following.

(c) *There exists a constant M_1 so that A is the generator of a k-times integrated semigroup $\{W(t)\}_{t\geq 0}$ such that*

$$\|W(t)\| \leq M_1 t^k, \quad \forall t \geq 0.$$

We conclude with a simple sufficient condition for A generating a $(1-A)^{-2}$-regularized semigroup that is $O((1+t)^k)$.

THEOREM 4.10: *Suppose the open right half plane $\{z \in \mathbf{C}\colon \mathrm{Re}(z) > 0\} \subseteq \rho(A)$, $k \in \mathbf{N}$, and there exists a constant M such that*

$$\|(z-A)^{-1}\| \leq M(\mathrm{Re}(z)^{-1} + \mathrm{Re}(z)^{-k}),$$

whenever $\mathrm{Re}(z) > 0$. *Then* A *generates a norm continuous* $(1-A)^{-2}$*-regularized semigroup that is* $O((1+t)^k)$.

Proof: Define, for $\epsilon > 0$,

$$W(t) \equiv \int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{zt}(z-A)^{-1} \frac{dz}{2\pi i(1-z)^2} \quad (t \geq 0).$$

By a calculus of residues argument, $W(t)$ is independent of $\epsilon > 0$. Dominated convergence implies that $t \mapsto W(t)$, from $[0, \infty)$ into $B(X)$, is continuous.

This construction is a special case of the functional calculus construction in [5, Chapter XXII]; the functional calculus properties in [5, Corollary 22.12] imply that $W(0) = (1-A)^{-2}$ and $W(t)W(s) = W(s+t)(1-A)^{-2}$, for any $s, t \geq 0$; that is, $\{W(t)\}_{t\geq 0}$ is a $(1-A)^{-2}$-regularized semigroup. By [5, Theorem 22.10(e)], A is the generator of $\{W(t)\}_{t\geq 0}$.

We will now verify the growth condition.

For $0 < \epsilon < 1$,

$$\|W(t)\| \leq M \int_{\mathbf{R}} e^{\epsilon t}(\epsilon^{-1} + \epsilon^{-k}) \frac{dy}{2\pi(1+|y|)^2} = \frac{M}{2} e^{\epsilon t}(\epsilon^{-1} + \epsilon^{-k}) \leq M e^{\epsilon t}\epsilon^{-k}.$$

For $t > 1$, let $\epsilon \equiv 1/t$, to conclude that

$$\|W(t)\| \leq (Me)t^k, \quad \forall t > 1.$$

This implies that $\|W(t)\|$ is $O((1+t)^k)$. ∎

Remarks 4.11: Note that Theorem 4.10 is guaranteeing mild $O((1+t)^k)$ solutions of (ACP), for all initial data $x \in \mathcal{D}(A^2)$.

If A generates a $(1-A)^{-2}$-regularized semigroup that is $O((1+t)^k)$, then, by Proposition 4.3, there exists a constant M so that

$$\|(z-A)^{-1}(1-A)^{-2}\| \leq M(\mathrm{Re}(z)^{-1} + \mathrm{Re}(z)^{-(k+1)}),$$

whenever $\mathrm{Re}(z) > 0$.

Thus the sufficient condition of Theorem 4.10 is "close" to a necessary condition.

If $\{\|s(s-A)^{-1}\|: s > 0\}$ were bounded, so that we could define fractional powers in the usual way (see [9] or [18]), then under the hypotheses of Theorem 4.10, we could define, with a little more work, as in [4, Section IV], for any $r > 0$, a $(1-A)^{-(1+r)}$-regularized semigroup generated by A that is $O((1+t)^k)$; this produces mild solutions of (ACP) for any initial data $x \in \mathcal{D}(A^{1+r})$.

A sufficient condition similar to that of Theorem 4.10, for generating a certain class of m-times integrated semigroups that includes those in Theorem 4.9, may be found in [20].

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